

# Flow Calibrated Reinforcement Learning with Application in Transition Path Sampling

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## Abstract

This paper aims to develop a comprehensive method for integrating generative flow networks (GFlowNet) into reinforcement learning (RL) for stochastic control issues. Our primary goal is to use GFlowNet to mitigate the impact of terminal cost during the initial phases of RL training, a process we term "flow calibration." Specifically, we concentrate on creating a flow-calibrated soft actor-critic (SAC) algorithm to address the transition path sampling (TPS) problem, which focuses on sampling rare transitions between two metastable states. TPS can be framed as a stochastic control problem and is generally challenging to train with traditional RL techniques in high-dimensional spaces due to the significant influence of terminal cost. Furthermore, to prevent discretization errors, we propose a continuous-time formulation of SAC and GFlowNet, grounded in the stochastic optimal control matching and Girsanov theorem. The central element linking the two models is the quantization strategy, and we also employ principal bundle theory to further utilize physics informations. Numerical results of generated transition paths for the Müller potential and Alanine dipeptide validate the effectiveness of these methods in both data-rich and data-scarce scenarios.

**Keywords** stochastic optimal control, reinforcement learning, generative flow network, transition path sampling, rare event sampling

## 1 Introduction

Rare event sampling constitutes a multidisciplinary field with applications spanning finance [14], molecular dynamics[31], and beyond. These rare events often arise due to the metastable behavior exhibited by the dynamical systems of interest. Metastability implies that the system remains confined to specific closed regions of its state space for extended periods, rarely making transitions to other closed regions. Precise sampling of these rare events necessitates a deep understanding of this transition behavior.

Remarkably, the average waiting time for a rare event significantly exceeds the intrinsic timescale of the underlying process. This phenomenon commonly manifests itself in dynamical systems governed by Langevin dynamics and navigating potentials with multiple minima. In such scenarios, the metastable closed regions align with local minima in the potential landscape. These minima are separated by energy barriers, and the transitions between them carry crucial information such as the macroscopic properties of the studied molecules. Furthermore, empirical evidence confirms that the time required to overcome these barriers scales exponentially with their height [4]. From a sampling perspective, the variance of Monte Carlo estimators associated with these rare transitions tends to be very high.

Many efforts have been devoted to the two issues presented above. The most notable methods can be classified into two groups[24]:

- **Splitting Method:** This idea involves breaking down a rare event into a series of moderately rare events, making it more tractable to sample. If only the transition between to local minima points is concerned, this procedure can be achieved adaptively using Adaptive Multilevel Splitting algorithm, which concurrently evolves an ensemble of trajectories and eliminates those who lags behind, whilst replicates that successfully explore the paths towards the target states; see [8, 2, 6, 7].

- **Importance Sampling:** This method biases the dynamics (e.g. modifying the potential) to reduce the variance, as detailed in [13, 5] and [23, Section 6.2]. If both the starting point and the target point are given, one can formulate the problem as a Schrodinger Bridge Matching problem[17] or it can be solved via an OT scheme[11]. However, to cope with more general transitions between two metastable states where the target point is not given, one may need to choose from adapting Stochastic Optimal Control formulation[40](fixed starting point) , approximating the target distribution(unknown) via a neural network[36](need sample paths) or sampling in a latent space [30] by means of Boltzmann generator(very slow). To our knowledge, none of the aforementioned methods can efficiently deal with transitions between metastable states, which are essentially two closed regions without conditioning on either the starting point or target point.

As a fundamental machine learning paradigm with notable achievements in complex, high-dimensional tasks such as Go [34, 35], recent initiatives have explored addressing classical Stochastic Optimal Control challenges, including the aforementioned one[24, 18], through a reinforcement learning (RL) lens, and is notably benefits large-scale Markov Decision Processes where traditional methods like dynamic programming are not viable. Yet, existing studies are either missing comprehensive theoretical backing [18] or are limited to discrete-time frameworks [32], both posing risks of errors in outcomes. Beyond issues of discretization, RL tends to produce a unimodal target, even when entropy regularization is employed [22]. This could lead to the accumulation of target points at local minima when a standard approximation of terminal costs is used. On the contrary, GFlowNet[3], a well-celebrated amortize sampler, is capable of generating multimodal ones, and this capability has been verified in various scenarios[19, 41]. Furthermore, when the search space is large and the time horizon is long, RL algorithms may require an extended period to converge, as only the final step’s update incorporates terminal costs. This classic dilemma of exploration-exploitation is always a core topic in RL research, such as [37, 33]. In comparison, GFlowNet equipped with trajectory balance loss[28] transmits the feedback of terminal cost to early sampling steps on each training step.

To address these challenges, we undertake the following steps: Initially, leveraging a ready-made theoretical result from transition path theory[15], we develop a novel formulation suitable for creating pathways between two metastable states, specifically from a known initial distribution to an unspecified target distribution, instead of merely from point-to-point or point-to-distribution. Subsequently, we introduce continuous-time versions of SAC and GFlowNet, with their sampling policies represented by parameterized SDEs, utilizing the stochastic optimal control matching[10] and Girsanov theorem respectively to eliminate discretization errors. To resolve the exploration-exploitation trade-off, we introduce a ‘flow calibration’ strategy that merges the immediate feedback benefits of RL with the foresight of GFlowNet, grounded by a quantization scheme, and further incorporates physics information via principal bundle theory, utilizing the idea from equivariant flow matching[21]. Notably, while our focus is on the TPS problem setting, our approaches are readily adaptable to broader SOC issues, a topic we will revisit in the discussion section.

Following this thread, the paper is structured as follows. In Section 2 we set the stage of rare event simulation, present the transition path sampling problem in section 2.1 and derive its stochastic optimal control formulation for distribution to distribution case in 2.2. Section 3 is dedicated to the introduction of the RL and GFlowNet framework. In Section 3.1, we discuss the underlying theoretical framework of reinforcement learning in SOC . In Section 3.2 we will briefly recap the key ideas of Soft Actor-Critic(SAC) algorithm[16] employed as RL part of our design. As for section 3.3, we introduce the GFlowNet for SOC, namely CFlowNet[25], under the same frame. Section 4 is devoted to continuous time generalization for both algorithms. In Section 4.1 we show how to represent the strategy of both algorithms via SDEs. More precisely, a single forward SDE for SAC and coupled SDEs for GFlowNet. In Section 4.2 we derive the continuous version of SAC from HJB equations following recent advances in continuous time RL[20]. In Section 4.3 we analyze the limit of trajectory balance loss[28] and derive its continuous time adaptation from Girsanov theorem. In Section 5, we demonstrate our "physics-informed reparameterization" starting with an introduction to equivariant flow in 5.1, followed by an exploration of the connection between reparameterization and variance in 5.2, and finally, we describe our novel reparameterization approach based on principal bundle theory in 5.3. In Section 6, we provide a brief overview of stochastic quantization in 6.1 and develop our final strategy using this concept in 6.2, which seamlessly integrates the different representations of SAC and GFlowNet strategies. In Section 7, we showcase our experimental results, highlighting their superiority over existing TPS algorithms. Finally, we conclude the paper with a discussion of our findings.

## 2 Stochastic Optimal Control for Transition Path Sampling

In this section, we will develop a distribution-to-distribution formulation for transition path sampling problem. First, we bias the potential landscape, which changes the drift term of SDE representing original molecule dynamics, to reweigh the sampling process. Second, we show that the distribution to distribution case can be reduced to the point to distribution case and derive that the bias potential term in point to distribution case can be solved from a stochastic control problem.

### 2.1 Bias Potential Method

Following the settings in [36], we will derive the bias potential method for transition path sampling in this subsection. Notice, that we will assume that  $X_t$  always has a fixed starting point in this section.

#### 2.1.1 Definition of Transition Paths

Given probability space  $(\Omega, \Sigma, \rho)$ . Suppose  $V(x) : \mathbb{R}^d \rightarrow \mathbb{R}$  be the potential of the system,  $W_t(\omega) : [0, +\infty) \times \Omega \rightarrow \mathbb{R}^d$  be the d-dimensional Brownian motion w.r.t. measure  $\rho$ . Let two metastable states be represented by closed closed regions  $A, B \subset \mathbb{R}^d$ . Their respective boundaries are  $\partial A, \partial B$ . In studying transition paths, it is often useful to model the governing system as an SDE. In this section, we will look at the overdamped Langevin equation, given by

$$dX_t = -\nabla V(X_t)dt + \sqrt{2\beta}dW_t, \quad X_0 = x, \quad (1)$$

where  $\beta$  is the diffusion coefficient. We omit the matrix  $I_d$  in the SDE above and the same hereafter, if the context is clear.

Let  $\mathcal{F}_t$  represent the filtration generated by the process  $X_t$  up to time  $t$ . Assume the path measure of  $X_t$  is  $\mathbb{P}^x$ , its marginals at stopping time  $\tau$  being  $\mathbb{P}_\tau^x$  respectively.

**Assumption 2.1.** Assume  $V$  is regular enough such that the invariant probability distribution exists

More precisely, if it exists, then it is given by

$$p(x) = e^{-\beta^{-1}V(x)}/Z, \quad Z = \int_{\Omega} e^{-\beta^{-1}V(x)}dx. \quad (2)$$

**Remark 2.2.** In transition path theory(TPT), a commonly seen term is "collective variable", which are functions of atomic coordinates used for dimension reduction. The case we considered above is the formulation under collective variable, because the Langevin dynamics is overdamped and can be represented by one SDE, instead of a couple of SDEs. In fact, the original Langevin dynamics without collective variable is given by

$$\begin{cases} dX_t &= m^{-1}P_t dt \\ dP_t &= [-\gamma P_t - \nabla V(X_t)] dt + \sqrt{2\gamma m \epsilon} dW_t, \end{cases} \quad (3)$$

and its invariant probability distribution is given by

$$\phi(x, p) = Z_H e^{-H(x,p)/\epsilon}, \quad H(x, p) = \frac{p^2}{2m} + V(x), \quad Z_H = \int_{\Omega} e^{-H(x,p)/\epsilon} dx dp. \quad (4)$$

It can be shown that this collective-variable-free case also has an SOC formulation[40], and can be solved via Path Integral Cross Entropy method[17]. There is no inherent difficulty to extend our algorithm to the collective variable free case, but we will focus on collective variable case for simplicity.

**Definition 2.3.** The paths that go from the boundary of  $A$  to the boundary of  $B$  without returning to  $A$  are called **transition paths** [12]. Mathematically, a realization  $(X_s)_{s=0}^T$  is a transition path if  $X_0 \in \partial A, X_T \in \partial B$ , and  $X_s \notin A \cup B, \forall s \in (0, T)$ .

#### 2.1.2 Distribution of Transition Paths

We will now examine the distribution of transition paths as in [27].

**Definition 2.4.** Given time  $t$ , the **hitting time of the process**  $X_t$  w.r.t. filtration  $\mathcal{F}_\infty$  starting at time  $s$  and position  $x$  is defined as:

$$\tau_A^{(X)} := \inf\{t \geq s : X_t \in A\}, \quad (5)$$

$$\tau_B^{(X)} := \inf\{t \geq s : X_t \in B\}. \quad (6)$$

We may omit the superscript  $(X)$  sometimes if the context is clear.

**Assumption 2.5.** For given  $X_s$  starting at starting point  $x$ , we always assume the system is ergodic, that is  $\tau_A^{(X)}, \tau_B^{(X)} < +\infty$ .

**Remark 2.6.** This condition can be slightly relaxed to  $\tau_A^{(X)}, \tau_B^{(X)} < +\infty$  almost surely with respect to  $\mathbb{P}^x$ , the path measure of  $X_t$ [36].

From this definition, we introduce a key ingredient for solving TPS problem named committor

**Definition 2.7.** Let  $E$  be the event that  $\tau_B^{(X)} < \tau_A^{(X)}$ . Given starting point  $x$ , the **committor** is defined as

$$q(x) := \mathbb{P}^x(E) \quad (7)$$

That is, the committor gives the probability that a path starting from a particular point  $x$  reaches closed region  $B$  before closed region  $A$ .

**Definition 2.8.** We call a stopping time

$$\tau_{AB}^{(X)} := \inf\{t \geq 0 | X_t \in A \cup B\} = \tau_A^{(X)} \wedge \tau_B^{(X)} \quad (8)$$

the **reactive time** of path  $X_t$ .

**Remark 2.9.** From the ergodic assumption, we have  $\tau_{AB} < +\infty$ , that is, the reactive time is always finite.

Before we give the SDE of the transition path, we need the following lemma

**Lemma 2.10.** The committor  $q$  is determined by the following boundary value problem

$$\begin{cases} \mathcal{L}q(x) = 0 & \text{if } x \notin A \cup B \\ q(x) = 0 & \text{if } x \in A \\ q(x) = 1 & \text{if } x \in B \end{cases}, \quad (9)$$

where  $\mathcal{L}$  is the generator of original diffusion(time-homogeneous), given by

$$\mathcal{L} = -\nabla V(X_t)\nabla + \beta\Delta \quad (10)$$

*Proof.* Suppose  $h(x)$  is a solution of the boundary value problem.

For  $x \in A \cup B$ , apparently,  $h$  coincides with  $q$ , which is  $1_B$ .

For  $x \in (A \cup B)^c$ , from [29, Theorem 9.2.13], we have

$$h(x) = \mathbb{E}_{\mathbb{P}^x}[1_B(X_{\tau_{AB}})] \quad (11)$$

$$= \mathbb{P}^x(\tau_A < \tau_B) \quad (12)$$

$$= \mathbb{P}^x(E) \quad (13)$$

$$= q(x) \quad (14)$$

Hence,  $h \equiv q$  for any  $x$ . Refer to [38] for more details.  $\square$

We claim that

**Theorem 2.11.** The dynamics of transition paths starting at  $x$  is given by

$$dY_t = \left(-\nabla V(Y_t) - 2\beta \frac{\nabla q(Y_t)}{q(Y_t)}\right)dt + \sqrt{2\beta}dW_t, \quad Y_0 = x \in A^c \quad (15)$$

*Proof.* We will use Doob's h-transform, a common technique for conditional dynamic to derive the equation. Define h-function

$$h(t, x) := \mathbb{P}(\tau_B^{(X)} < \tau_A^{(X)} | X_t = x) \quad (16)$$

Denote the generator of transition paths as  $\mathcal{G}^{(t)}$ , i.e.

$$\mathcal{G}^{(t)} f(x) = \lim_{s \rightarrow 0} \frac{1}{s} \left( \mathbb{E} \left[ f(X_{t+s}) | X_t = x, \tau_B^{(X)} < \tau_A^{(X)} \right] - f(x) \right) \quad (17)$$

$$= \lim_{s \rightarrow 0} \frac{\mathbb{E}[f(X_{t+s})h(t+s, X_{t+s}) | X_t = x] - f(x)h(t, x)}{sh(t, x)} \quad (18)$$

$$= \lim_{s \rightarrow 0} \frac{\mathbb{E}[f(X_{t+s})q(X_{t+s}) | X_t = x] - f(x)q(x)}{sq(x)} \quad (19)$$

for arbitrary function  $f$ . From the definition above, the generator of biased diffusion (can be non-time-homogeneous in general Doob's h-transform) is given by

$$\mathcal{G}^{(t)} f = \frac{1}{q} (\partial_t + \mathcal{L})(qf) \quad (20)$$

$$= \frac{1}{q} \mathcal{L}(qf), \quad (21)$$

because  $q(x)f(x)$  apparently does not depend on time in our situation. From previous Lemma 2.10

$$\mathcal{L}q = 0 \quad (22)$$

Hence

$$\mathcal{G}^{(t)} f = \mathcal{L}f + \frac{2\beta \nabla q}{q} \cdot \nabla f \quad (23)$$

which completes our proof. See [9] for more details.  $\square$

**Remark 2.12.** *In fact, the theorem still holds for  $x \in \partial A$ , where the committor  $q$  vanishes, that is, this SDE still admits a unique strong solution. See [27, Theorem 1.1] for the proof.*

## 2.2 Solving Bias Potential from Stochastic Optimal Control

In this subsection, we begin with reducing the distribution to distribution case to point to distribution case, and then derive a SOC formulation from minimal action in optimal transport.

### 2.2.1 Reduction of initial condition

Consider Overdamped Langevin Dynamics with given initial distribution  $\mu$ , i.e.

$$dX_t = -\nabla V(X_t)dt + \sqrt{2\beta}dW_t, \quad X_0 \sim \mu. \quad (24)$$

Denote its corresponding path measure as  $\mathbb{P}^\mu$ , and its marginal at stopping time  $\tau$  as  $\mathbb{P}_\tau^\mu$ .

**Definition 2.13.** *The relative entropy between two measures  $\alpha, \beta$  is given by*

$$Ent(\alpha|\beta) := \begin{cases} \int_\zeta \log \left( \frac{d\alpha}{d\beta} \right) d\alpha & \text{if } \alpha \ll \beta, \\ +\infty & \text{otherwise,} \end{cases} \quad (25)$$

for measurable space  $\zeta$ .

The next theorem is related to the Schrodinger Bridge formulation of the general sampling problem, whose connection with TPS will be discussed later. We will use the proof in [15, Theorem 4.3]

**Theorem 2.14.** *Suppose the strictly convex terminal cost  $g : \mathbb{R}^d \rightarrow [0, +\infty]$  is  $\mathcal{F}_\tau$  measurable, there exists a  $\mathcal{F}_\tau$  measurable set  $D \subset \text{dom } g$ , s.t.  $\mathbb{P}_\tau^x(D) > 0$ , and that  $\int_D g d\mathbb{P}_\tau < +\infty$ .*

*Then, the following holds:*

1. The point-to-distribution sampling problem

$$\inf_{\mathbb{Q}} \{ \mathbb{E}_{\mathbb{Q}}[g(X_{\tau}) + \log \frac{d\mathbb{Q}}{d\mathbb{P}^x}] | \mathbb{Q} \in \mathcal{P}(\Omega) \}, \quad (26)$$

where  $\mathcal{P}(\cdot)$  are all measures on certain sample space, admits unique optimal path measure  $\mathbb{Q}^{*,x}$ , satisfying  $\mathbb{Q}_0^{*,x} = \mu$

2. The distribution-to-distribution sampling problem

$$\inf_{\mathbb{Q}} \{ \mathbb{E}_{\mathbb{Q}}[g(X_{\tau}) + \log \frac{d\mathbb{Q}}{d\mathbb{P}^{\tilde{\mu}}}] | \mathbb{Q} \in \mathcal{P}(\Omega), \mathbb{Q}_0 = \tilde{\mu} \}, \quad (27)$$

where  $\tilde{\mu} \in \mathcal{P}(\mathbb{R}^d)$  is a given measure (not necessarily equals to  $\mu$ ) with  $\text{Ent}(\tilde{\mu}|\mu) < +\infty$ , has a minimizer taking the form  $\mathbb{Q}^{*,\tilde{\mu}} = \tilde{\mu} \otimes \mathbb{Q}^{*,x}$  (the definition of  $\otimes$  is in the proof)

**Remark 2.15.** In our case,  $\tilde{\mu} = \mu$ . It indicates that we only need to solve the measure  $\mathbb{Q}^{*,x}$  for the point-to-distribution case on one point  $x$ . For the distribution-to-distribution case, it suffices to randomize the starting point  $x$  by  $x \sim \mu$  and "shoot" with the same conditional measure  $\mathbb{Q}^{*,x}$ .

*Proof.* We will give a proof for 1 and 2 for completeness.

1. First, we show that the set of admissible path measures is nonempty, that is, there exists at least one path measure, such that the stochastic control problem in 1 has a finite cost. To construct such a measure, we reweight the probability of  $\mathbb{P}_{\tau}^x$  to make its support set be  $D$ , i.e.

$$\nu := \frac{1_D \mathbb{P}_{\tau}^x}{\mathbb{P}_{\tau}^x(D)}. \quad (28)$$

Make a path measure  $\mathbb{Q}$  to have marginal  $\nu$  at stopping time  $\tau$ , that is

$$\mathbb{Q} := \nu \otimes \mathbb{P}_{|X_{\tau}}^x = \int_{\mathbb{R}^d} \mathbb{P}_{|X_{\tau}=\eta}^x d\nu(\eta), \quad (29)$$

where  $\mathbb{P}_{|X_{\tau}=\eta}^x$  is measure  $\mathbb{P}^x$  conditioning on  $X_{\tau} = \eta$ .

We claim this measure is admissible, verified as follows

$$\mathbb{E}_{\mathbb{Q}}[g(X_{\tau}) + \log \frac{d\mathbb{Q}}{d\mathbb{P}^x}] = \mathbb{E}_{\mathbb{Q}}[g(X_{\tau}) + \log \frac{d\mathbb{Q}_{\tau}}{d\mathbb{P}^x}(X_{\tau})], \text{ since it is the only difference between } \mathbb{Q} \text{ and } \mathbb{P}^x \quad (30)$$

$$= \mathbb{E}_{\nu}[g(x) + \log \frac{d\nu}{d\mathbb{P}^x}(x)] \quad (31)$$

$$= \int_{\mathbb{R}^d} g d\nu + \text{Ent}(\nu|\mathbb{P}_{\tau}^x), \text{ where Ent is the relative entropy} \quad (32)$$

$$= \frac{\int_D g d\mathbb{P}_{\tau}^x}{\mathbb{P}_{\tau}^x(D)} - \log \mathbb{P}_{\tau}^x(D) \quad (33)$$

$$< +\infty \quad (34)$$

Next, we move to the existence of minimizers by the Direct Method of the Calculus of Variations.

Step one, construct such a candidate minimizer. From previous argument, the minimizing sequence  $\mathbb{Q}^k$  exists, that is

$$\sup_{k \geq 1} \mathbb{E}_{\mathbb{Q}^k}[f(X_{\tau}) + \log \frac{d\mathbb{Q}^k}{d\mathbb{P}^x}] < +\infty \quad (35)$$

Since  $f \geq 0$ , it implies

$$\sup_{k \geq 1} \text{Ent}(\mathbb{Q}^k|\mathbb{P}^x) < +\infty \quad (36)$$

Thus, by de la Vallée Poussin theorem, the sequence  $(\frac{d\mathbb{Q}^k}{d\mathbb{P}^x})_{k \geq 1}$  is uniformly integrable. Further, since the sequence is bounded in  $L^1(\Omega, \mathbb{P}^x)$ , Dunford-Pettis theorem provides that this sequence is also

convergent weakly in  $L^1(\Omega, \mathbb{P}^x)$ . Denote the weak limit as  $Z$ , and let the candidate be  $\mathbb{Q} := Z\mathbb{P}^x$ , then we have convergence  $\mathbb{Q}^k \xrightarrow{\text{setwise}} \mathbb{Q}$ .

Step two, we show that  $\mathbb{Q}$  is indeed the minimizer we want

$$\mathbb{E}_{\mathbb{Q}}[(g \wedge n)(X_\tau) + \log \frac{d\mathbb{Q}}{d\mathbb{P}^x}] \leq \lim_{k \rightarrow \infty} \mathbb{E}_{\mathbb{Q}^k}[(g \wedge n)] + \liminf_{k \rightarrow \infty} Ent(\mathbb{Q}|\mathbb{P}^x), \text{ by lower semi-continuity} \quad (37)$$

$$\leq \liminf_{k \rightarrow \infty} [(g \wedge n)(X_\tau) + \log \frac{d\mathbb{Q}^k}{d\mathbb{P}^x}] \quad (38)$$

$$\leq \liminf_{k \rightarrow \infty} [g(X_\tau) + \log \frac{d\mathbb{Q}}{d\mathbb{P}^x}]. \quad (39)$$

Take  $n \rightarrow \infty$ . From the monotone convergence theorem, we have

$$\mathbb{E}_{\mathbb{Q}}[g(X_\tau) + \log \frac{d\mathbb{Q}}{d\mathbb{P}^x}] \leq \liminf_{k \rightarrow \infty} \mathbb{E}_{\mathbb{Q}^k}[g(X_\tau) + \log \frac{d\mathbb{Q}^k}{d\mathbb{P}^x}]. \quad (40)$$

The uniqueness of the minimizer  $\mathbb{Q}$  follows directly from the strict convexity of  $g$ .

2. Notice that for any  $\mathbb{Q} \in \mathcal{P}(\Omega)$ ,  $\mathbb{Q}_0 = \tilde{\mu}$ , we have:  $\mathbb{Q}_0 \ll \mu$  (Since  $Ent(\tilde{\mu}|\mu) < +\infty$ ). Moreover, by disintegration theorem, we have:

$$\frac{d\mathbb{Q}}{d\mathbb{P}^\mu}(\omega) = \frac{d\mathbb{Q}_0}{d\mathbb{P}_0^\mu}(X_0(\omega)) \frac{d\mathbb{Q}|_{X_0=x}}{d\mathbb{P}^\mu|_{X_0}}(\omega), \text{ for } \mathbb{Q}\text{-almost } \omega \in \Omega \quad (41)$$

This induces the disintegration of relative entropy, which writes:

$$Ent(\mathbb{Q}|\mathbb{P}^\mu) = Ent(\tilde{\mu}|\mu) + \int_{\mathbb{R}^d} Ent(\mathbb{Q}|_{X_0=x}|\mathbb{P}^x) \mathbb{Q}_0(dx) \quad (42)$$

The problem now becomes:

$$Ent(\tilde{\mu}|\mu) + \inf_{\mathbb{Q}} \left\{ \int_{\mathbb{R}^d} \mathbb{E}_{\mathbb{Q}|_{X_0=x}} [f(X_\tau) + \log \frac{d\mathbb{Q}|_{X_0=x}}{d\mathbb{P}^x}] \tilde{\mu}(dx) \right\} \quad (43)$$

Therefore, if  $\mathbb{Q}^{*,x}$  is the minimizer of the problem conditioning on initial point  $x$ , then the optimal path measure is given by  $\mathbb{Q}^{*,\tilde{\mu}} = \tilde{\mu} \otimes \mathbb{Q}^{*,x}$

□

Since the Theorem 2.11 and Remark 2.12 only holds for  $x \in (\text{int } A)^c$ , we need to clarify the movement of the particle starting at a point within  $A$  to make the boundary distribution of  $Y_t$  on  $\partial A$  coincides with the boundary distribution of  $X_t$ .

**Definition 2.16.** Suppose the initial distribution on  $A$  is given by  $\phi(x)$ . Define **reactive exiting distribution** of  $\partial A$  as  $\mu := \phi|_{\partial A}$ . Moreover, define the **empirical reactive exiting distribution** of  $\partial A$  as  $\mu_N := \frac{1}{N} \sum_{k=0}^{N-1} \delta_{X_{\tau_A, k}}(z)$ , where  $z \in A$  and  $k$  is the label for  $N$  iid samples.

Next, we extend the definition of  $Y_t$  to  $x \in A$  case simply by

$$Y_t := -\nabla V(Y_t)dt + \sqrt{2\beta}dW_t, 0 \leq t \leq \tau_A, x \in A \quad (44)$$

The following proposition can justify the definition

**Proposition 2.17.**  $\mu_N \xrightarrow{\text{weak}} \mu$  as  $N \rightarrow +\infty$ , i.e.

$$\lim_{N \rightarrow \infty} \int_{\partial A} f(x) d\mu_N(x) = \int_{\partial A} f(x) d\mu(x), \mathbb{P}^x - a.s. \quad (45)$$

for any continuous bounded function  $f : \partial A \rightarrow \mathbb{R}$ .

*Proof.* See [27, Proposition 1.4].

□

This proposition allows us to equalize the hitting distribution of  $Y_t$  on  $\partial A$  with  $\mu$ . Combining it Theorem 2.14, it suffices to construct SOC formulation for the point-to-distribution case with  $x \in \partial A$ .

### 2.2.2 Deriving SOC formulation

Define terminal cost as follows

$$g(x) = \begin{cases} 0, & x \in B \\ +\infty, & \text{otherwise} \end{cases} \quad (46)$$

Pick sufficiently small  $\epsilon > 0$ . Generate a starting point  $x \in A^c$  satisfying  $d(x, \partial A) < \epsilon$ , such as taking one more small step without bias after  $Y_t$  hits the boundary  $\partial A$  to go outside the closed region  $A$ .

Parameterize the path measure  $\mathbb{Q}$  by setting it as the law of the following SDE

$$dY_t = [-\nabla V(Y_t) + 2\beta v_F^\theta(Y_t)] dt + \sqrt{2\beta} dW_t, \quad Y_0 = x \in A^c. \quad (47)$$

**Remark 2.18.** *Different from the next section, we temporarily assume that  $\theta$  parameterizes  $v_F^\theta$ , a deterministic mapping from  $\mathbb{R}^d$  to  $\mathbb{R}^d$  here.*

To apply Girsanov's theorem to the entropy term above, we need

**Assumption 2.19.**  $\mathbb{E}_{\mathbb{Q}} \log \frac{d\mathbb{Q}}{d\mathbb{P}^x} < +\infty \iff \mathbb{E}_{\mathbb{Q}}[\int_0^{\tau_{AB}} |v_F^\theta(Y_t)|^2 dt] < +\infty$  (Novikov condition, derived from the following lines),

which gives

$$\mathbb{E}_{\mathbb{Q}}[\log \frac{d\mathbb{Q}}{d\mathbb{P}^x}] = \mathbb{E}_{\mathbb{Q}}[-\log \frac{d\mathbb{P}^x}{d\mathbb{Q}}] \quad (48)$$

$$= \mathbb{E}_{\mathbb{Q}}[\frac{1}{\sqrt{2\beta}} \int_0^{\tau_{AB}} v_F^\theta(Y_t) d\check{W}_t + \frac{1}{4\beta} \int_0^{\tau_{AB}} |v_F^\theta(Y_t)|^2 dt] \quad (49)$$

$$= \frac{1}{4\beta} \mathbb{E}_{\mathbb{Q}} \int_0^{\tau_{AB}} |v_F^\theta(Y_t)|^2 dt, \quad (50)$$

where we denote the Brownian motion w.r.t.  $\mathbb{Q}$  as  $\check{W}_t$  temporarily. Thus the original SOC problem is equivalent to

$$\inf_{\theta} \mathbb{E}_{\mathbb{Q}}[\frac{1}{4\beta} \int_0^{\tau_{AB}} |v_F^\theta(Y_t)|^2 dt + g(Y_{\tau_{AB}}) | Y_t \sim \mathbb{Q}, Y_0 = x]. \quad (51)$$

The next theorem shows that the stochastic optimal control problem presented above gives the transition path measure

**Corollary 2.20.** *For  $x \in A^c$ , the optimal control for the problem*

$$\inf_{\theta} \mathbb{E}_{\mathbb{Q}}[\frac{1}{4\beta} \int_0^{\tau_{AB}} |v_F^\theta(Y_t)|^2 dt + g(Y_{\tau_{AB}}) | Y_t \sim \mathbb{Q}, Y_0 = x], \quad (52)$$

is given by

$$v_F^{*\theta}(x) = \nabla \log q(x) = \frac{\nabla q(x)}{q(x)}. \quad (53)$$

That is, the optimal path measure for the original problem is the measure of the transition path  $Y_t$ .

*Proof.* See [40, Corollary 3.1.1]. □

Here is a brief summary of the procedure for our sampling

1. Choose a point on  $\partial A$ , and take a sufficiently small step following  $X_t$  to generate a starting point  $x \in A^c$
2. For this fixed  $x$ , solve the SOC problem in corollary 2.20 for  $v_F^{*\theta}(x)$
3. Pick any fixed point  $x_{in}$  in  $int A$  (normally the local minima)
4. Run the extended dynamic of  $Y_t$  (namely that of  $X_t$ ), until the path hits the boundary of  $A$  at  $x_{hit}$
5. Use the same SDE of  $v_F^{*\theta}$  to shoot from  $x_{hit}$
6. Repeat 4 to 5

Thus, we will focus on solving SOC problem for a fixed starting point in the following sections.

### 3 Discretized RL and GFN for Stochastic Optimal Control

#### 3.1 Discretized RL for stochastic optimal control

First, to avoid an infinite time horizon, pick a sufficiently large constant  $\tilde{T}$ , and set stopping time  $T = \tilde{T} \wedge \tau_{AB}$ .

Discretize the objective function and the dynamics with time step  $\Delta t$

$$\inf_{\theta} \mathbb{E}_{\hat{\mathbb{Q}}} \left[ \frac{1}{4\beta} \sum_{t=0}^{T-1} \Delta t |v_t|^2 + g(Y_T) | Y_0 = x \right]$$

s.t.  $Y_{t+\Delta t} = Y_t + [-\nabla V(Y_t) + 2\beta v_t] \Delta t + \sqrt{2\beta} \Delta W_t,$

where  $T$  is the time step when  $Y_{1:T-1} \in (A \cup B)^c$ ,  $\Delta W_t \sim \mathcal{N}(0, \Delta t)$ , and  $\hat{\mathbb{Q}}$  is the discretized adaptation of path measure  $\mathbb{Q}$ .

Second, we introduce regular elements in RL problem and clarify some concept in the equation above.

Let the state space  $\mathcal{S} := \mathbb{R}^d$ , the policy space  $\mathcal{P} := \{\pi_F^\theta | \pi_F^\theta : Y_t \in \mathbb{R}^d \rightarrow \pi_F^\theta(\cdot | Y_t) \in \mathcal{P}(\mathbb{R}^d)\}$ , namely all the mixed strategies on  $\mathbb{R}^d$  (we may choose different parameterized policy space  $\mathcal{P}_\theta$  according to the specific RL algorithm we use in practice). Denote the selected action at time  $t$  to be  $v_t \in \mathcal{A} = \mathbb{R}^d$ , where  $\mathcal{A}$  stands for action space.

**Remark 3.1.** *In fact, due to the deterministic  $\nabla \log q(x)$ , we should have used deterministic action space. However, in practice, the agent may benefit from training with stochastic strategy when the search space is very large, such as avoiding suboptimal policies, obtaining more robust learning outcomes, etc, especially for SAC algorithm[16], which is the case we will be concerned about.*

Define the intermediate reward to be  $r(v_t) := \frac{1}{4\beta} \Delta t |v_t|^2$ , and let the terminal reward be  $g(Y_T)$ . This formulation allows us to solve the discretized problem with RL, which is Soft Actor-Critic in our situation.

#### 3.2 Soft Actor Critic for Discretized SOC Problem

The key features of SAC algorithm are summarized as follows

1. Use entropy regularization to make the agent more explorative
2. Use clipped double  $Q(s, a)$  function to avoid overparameterization and overestimation
3. Use Gaussian policy, where the variance term is also parameterized, to make a bigger hypothesis set

Since we focus on continuous cases, we will not give a full introduction to the original SAC algorithm. Here is a complete pseudo-code implementation of SAC adapted from [1] for the discretized problem

---

**Algorithm 1** Soft Actor-Critic (SAC) algorithm for SOC problem

---

- 1: Initialize policy parameters  $\theta$ , Q-function parameters  $\phi_1, \phi_2$ , empty replay buffer  $\mathcal{D}$ , tradeoff constant  $\gamma$ , target parameter  $\phi_{\text{target},1} \leftarrow \phi_1, \phi_{\text{target},2} \leftarrow \phi_2$ , and smoothing parameter  $\rho$ .
- 2: **repeat**
- 3:   Observe state  $Y_t$  and select action  $v_t$  from distribution  $\pi_F^\theta(\cdot | Y_t)$ .
- 4:   Execute action  $v_t$  in the environment.
- 5:   Observe next state  $Y_{t+1}$ , reward  $r_{t+1}$ , and done signal  $d$  to indicate  $Y_{t+1} \in A \cup B$ .
- 6:   **if**  $d$  is True **then**  $r_{t+1} = r_{t+1} - g(Y_{t+1})$  and reset the environment state.
- 7:   Store transition  $(Y_t, v_t, r_{t+1}, Y_{t+1}, d)$  in replay buffer  $\mathcal{D}$ .
- 8:   **if** it's time to update **then**
- 9:     **for** each gradient step **do**
- 10:       Randomly sample a batch  $B$  of transitions  $(Y_t, v_t, r_{t+1}, Y_{t+1}, d)$  from  $\mathcal{D}$ .
- 11:       Compute targets for the Q functions
- 12:        $y_q = r_t + \gamma(1 - d) \min_{i=1,2} Q_{\phi_{\text{target},i}}(Y_{t+1}, v_{t+1}) - \gamma \log \pi_F^\theta(v_{t+1} | Y_{t+1}), \quad v_{t+1} \sim \pi_F^\theta(\cdot | Y_{t+1})$
- 13:       Update Q-functions by one step of gradient descent using
- 14:

$$\nabla_{\theta_i} \frac{1}{|B|} \sum_{(Y_t, v_t, r_{t+1}, Y_{t+1}, d) \in B} (Q_{\phi_i}(Y_t, v_t) - y_q)^2, \text{ for } i \in \{1, 2\}.$$

15: Update policy by one step of gradient ascent using

16:

$$\nabla_{\phi} \frac{1}{|B|} \sum_{Y_t \in B} \left( \min_{i=1,2} Q_{\phi_i}(Y_t, v_t) - \gamma \log \pi_F^{\theta}(v_t|Y_t) \right),$$

17: where the reparametrization trick is used, that is

$$v_t = \mu_{\theta}(Y_t) + \sigma_{\theta}(Y_t) \cdot \xi, \quad \xi \sim \mathcal{N}(0, I_d). \quad (54)$$

18: Update target network

$$\phi_{\text{targ},i} \leftarrow \rho \phi_{\text{targ},i} + (1 - \rho) \phi_i \quad \text{for } i = 1, 2 \quad (55)$$

19: **end for**

20: **end if**

21: **until** convergence

At test time, we will output the mean policy  $\mu_{\theta}$  to improve the behavior.

**Remark 3.2.** *Three key issues for the original SAC*

1. *Discretization error*
2. *Only consider terminal cost at the final step*
3. *may not generate very diverse trajectories or target points in sampling applications*

### 3.3 GFlowNet for Discretized SOC Problem

The key difference between GFlowNet and SAC is that it does not directly compute the reward at each intermediate step, instead, it collects all the reward at the final step and learns to construct consistent probability flow(both forward and backward) to match the distribution of the final reward, which is better regarding exploration ability[3] and is particularly suitable for sparse signal RL scenario.

Likewise, since we focus on continuous cases, we will only give the pseudo-code of it based on [25] and leave more space for continuous case

**Remark 3.3.** *Two key issues for GFN*

1. *Not very suitable for problems with intermediate cost[25]*
2. *Trajectory balance is very costly to train since they are evaluated on the entire trajectory at a time compared to SAC, especially when  $T$  is large.*

## 4 Continuous Generalization of SAC and GFN

### 4.1 Representation of Continuous Strategies via SDEs

#### 4.1.1 Representation of SAC Strategy via Forward SDE

Since the transition dynamic  $Y_t$  for SAC is already determined  $v_t \sim \pi_F^{\theta}(\cdot|Y_t)$ , a feedback stochastic control policy, the SAC strategy can be simply given by

$$v_t = \mu_{\theta}(Y_t) + \sigma_{\theta}(Y_t) \cdot \xi, \quad \xi \sim \mathcal{N}(0, \sqrt{h}I_d) \quad (57)$$

$$dY_t = [-\nabla V(Y_t) + 2\beta v_t] dt + \sqrt{2\beta} dW_t \quad (58)$$

where  $\mu_{\theta}, \sigma_{\theta}$  are some suitable NNs, and  $h$  being some suitable discretization step size. The unique strong solution does exist for arbitrary fixed  $(v_t)_{t=0}^T$  from [39, Chapter 1, Theorem 6.16] under mild assumptions, as long as we assume  $g$  is already approximated by some smooth functions, which is usually the case in practice[40, 18]. More rigorously, we need to write the variables as  $v_t^{\pi}, Y_t^{\pi}$ , but we will stick to the original notation if the context is clear, for consistency.

**Remark 4.1.** *In this light, we are dealing with a model-based case in TPS problem. However, our method can be extended to model-free cases without inherent difficulty.*

---

**Algorithm 2** Generative Flow Network (GFlowNet) algorithm for SOC Problem
 

---

**Initialize:** Policy network  $\theta$  for  $\pi_F^\theta$ , a pre-trained retrieval network  $\Upsilon$  to find parent, empty buffer  $\mathcal{D}$  and  $\mathcal{P}$ , clip constant  $\epsilon$ , reward weight  $\lambda$

**repeat**  
 2: Set  $t = 0, Y_0 = x$   
   **while**  $Y_t \neq \text{terminal}$  and  $t < \tilde{T}$  **do**  
 4:   Sample  $M$  actions  $v_t$  from action space  $\mathcal{A}$  according to  $\pi_F^\theta(\cdot|Y_t)$   
   Execute  $v_t$  in the environment to obtain  $r_{t+1}$  and  $s_{t+1}$   
 6:    $t = t + 1$   
**end while**  
 8: Store episodes  $\{(Y_t, v_t, r_{t+1}, Y_{t+1})\}_{t=1}^T$  in replay buffer  $\mathcal{D}$   
 [Optional] Fine-tuning retrieval network  $G_\Upsilon$  based on  $\mathcal{D}$   
 10: Sample a random batch  $B$  of episodes from  $\mathcal{D}$   
 Uniformly sample  $K$  actions  $\{v_t^k\}_{k=1}^K$  from action space  $\mathcal{A}$  for each state in  $B$   
 12: Compute parent states according to  $\{G_\Upsilon(Y_t^k, v_t^k)\}_{k=1}^K$  for each state in  $B$   
 Update  $\pi_F^\theta$  according to

$$\mathcal{L}_\theta(\tau) = \sum_{t=1}^T \left\{ \log \left[ \epsilon + \sum_{k=1}^K \hat{\mathbb{Q}}(G_\Upsilon(Y_t^k, v_t^k) \xrightarrow{v_t^k} Y_t^k) \right] - \log \left[ \epsilon + \sum_{k=1}^K (\hat{\mathbb{Q}}(Y_t^k \xrightarrow{v_t^k} Y_{t+1}^k) + \lambda(r(v_t) - 1_{t=T}g(Y_T))) \right] \right\}^2, \quad (56)$$

14: **until** convergence

---

#### 4.1.2 Representation of GFlowNet Strategy via coupled SDEs

Next, we will derive the coupled SDE for GFlowNet. Let us establish the theorem for a more general case. Notice, that all the  $\nabla, \Delta, \text{div}$  is only taken to the spatial component in the following theorem and its proof.

**Theorem 4.2.** *Suppose we are given a general diffusion process*

$$dX_t = f(X_t, t)dt + \sigma(X_t, t)dW_t. \quad (59)$$

where  $f : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^{d+1}, \sigma : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^{(d+1) \times (d+1)}$ . Its time reversal is given by the following equation:

$$dX_{\tilde{t}} = [f(X_{\tilde{t}}, \tilde{t}) - \text{div}(\sigma(X_{\tilde{t}}, \tilde{t})\sigma^T(X_{\tilde{t}}, \tilde{t})) \quad (60)$$

$$- \sigma(X_{\tilde{t}}, \tilde{t})\sigma^T(X_{\tilde{t}}, \tilde{t})\nabla \log p(X_{\tilde{t}}, \tilde{t})]d\tilde{t} + \sigma(X_{\tilde{t}}, \tilde{t})d\tilde{W}_{\tilde{t}}. \quad (61)$$

Here, we denote the time with a negative increment with  $\tilde{t}$  and the backward Brownian motion with  $\tilde{W}_{\tilde{t}}$ .

*Proof.* Write the forward FPK equation

$$\frac{\partial p_t}{\partial t}(X_t, t) = -\text{div} \left\{ f(X_t, t)p(X_t, t) - \frac{1}{2} \text{div}[\sigma(X_t, t)\sigma^T(X_t, t)p(X_t, t)] \right\}. \quad (62)$$

Notice

$$\text{div}[\sigma(X_t, t)\sigma^T(X_t, t)p(X_t, t)] \quad (63)$$

$$= p(X_t, t)\text{div}[\sigma(X_t, t)\sigma^T(X_t, t)] + \sigma(X_t, t)\sigma^T(X_t, t)\nabla p(X_t, t) \quad (64)$$

$$= p(X_t, t)\text{div}[\sigma(X_t, t)\sigma^T(X_t, t)] + p(X_t, t)\sigma(X_t, t)\sigma^T(X_t, t)\nabla \log p(X_t, t). \quad (65)$$

We have

$$\frac{\partial p}{\partial t}(X_t, t) = -\text{div}\{[f(X_t, t) \quad (66)$$

$$- \frac{1}{2} \text{div}(\sigma(X_t, t)\sigma^T(X_t, t)) \quad (67)$$

$$- \frac{1}{2} \sigma(X_t, t)\sigma^T(X_t, t)\nabla \log p(X_t, t)] \cdot p(X_t, t)\}. \quad (68)$$

It corresponds to forward ODE

$$dX_t = [f(X_t, t) - \frac{1}{2} \operatorname{div}(\sigma(X_t, t)\sigma^T(X_t, t)) \quad (69)$$

$$- \frac{1}{2} \sigma(X_t, t)\sigma^T(X_t, t)\nabla \log p(X_t, t)]dt. \quad (70)$$

It gives the reverse ODE with a positive time increment ( $t$  starts from the beginning)

$$dX_t = - [f(X_t, t) - \frac{1}{2} \operatorname{div}(\sigma(X_t, t)\sigma^T(X_t, t)) \quad (71)$$

$$- \frac{1}{2} \sigma(X_t, t)\sigma^T(X_t, t)\nabla \log p(X_t, t)]dt. \quad (72)$$

The FPK equation for reverse ODE with a positive time increment is

$$\frac{\partial p}{\partial t}(X_t, t) = - \operatorname{div}\{ - [f(X_t, t) - \frac{1}{2} \operatorname{div}(\sigma(X_t, t)\sigma^T(X_t, t)) \quad (73)$$

$$- \frac{1}{2} \sigma(X_t, t)\sigma^T(X_t, t)\nabla \log p(X_t, t)] \cdot p(X_t, t)\} \quad (74)$$

$$= - \operatorname{div}\{ - [f(X_t, t) - \operatorname{div}(\sigma(X_t, t)\sigma^T(X_t, t)) \quad (75)$$

$$- \sigma(X_t, t)\sigma^T(X_t, t)\nabla \log p(X_t, t)] \cdot p(X_t, t)\} \quad (76)$$

$$+ \frac{1}{2} \Delta(\sigma(X_t, t)\sigma^T(X_t, t)p(X_t, t)). \quad (77)$$

Thus, the corresponding SDE with a positive time increment is

$$dX_t = - [f(X_t, t) - \operatorname{div}(\sigma(X_t, t)\sigma^T(X_t, t)) \quad (78)$$

$$- \sigma(X_t, t)\sigma^T(X_t, t)\nabla \log p(X_t, t)]dt + \sigma(X_t, t)dW_t. \quad (79)$$

Reverse the time, we have SDE with a negative time increment ( $\tilde{t}$  starts from the end)

$$dX_{\tilde{t}} = [f(X_{\tilde{t}}, \tilde{t}) - \operatorname{div}(\sigma(X_{\tilde{t}}, \tilde{t})\sigma^T(X_{\tilde{t}}, \tilde{t})) \quad (80)$$

$$- \sigma(X_{\tilde{t}}, \tilde{t})\sigma^T(X_{\tilde{t}}, \tilde{t})\nabla \log p(X_{\tilde{t}}, \tilde{t})]d\tilde{t} + \sigma(X_{\tilde{t}}, \tilde{t})d\check{W}_{\tilde{t}}, \quad (81)$$

as desired.  $\square$

Unlike SAC, we assume the strategy  $v_t$  is also evolving in an SDE form, that is

$$dv_t = \mu_\theta(Y_t)dt + \sigma_\theta(Y_t)d\check{W}_t, \quad (82)$$

where  $\check{W}_t$  is the noise independent of  $W_t$ ,  $\mu_\theta : \mathbb{R}^d \rightarrow \mathbb{R}^d$  is the mean strategy and  $\sigma_\theta : \mathbb{R}^d \rightarrow \mathbb{R}$  is the noise level of the strategy.

**Remark 4.3.** We can choose  $\sigma_\theta : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$  if the world model is very complex. However, this is not likely the case in TPS, since the dynamics are explicitly given. We will only use a parameterized constant to control the noise level, which, to some extent, presumes an isotropy of the strategy.

We want to derive an explicit reverse SDE system corresponding to

$$\begin{cases} dY_t = [-\nabla V(Y_t) + 2\beta v_t]dt + \sqrt{2\beta}dW_t \\ dv_t = \mu_\theta(Y_t)dt + \sigma_\theta(Y_t)d\check{W}_t, \end{cases} \quad (83)$$

which can be rewritten as

$$d \begin{pmatrix} Y_t \\ v_t \end{pmatrix} = \begin{pmatrix} -\nabla V(Y_t) + 2\beta v_t \\ \mu_\theta(Y_t) \end{pmatrix} dt + \begin{pmatrix} \sqrt{2\beta}I_d & 0 \\ 0 & \sigma_\theta(Y_t) \end{pmatrix} \cdot \begin{pmatrix} dW_t \\ d\check{W}_t \end{pmatrix}. \quad (84)$$

Applying theorem 4.2, we immediately have

**Corollary 4.4.** The reverse representation of the system is given by

$$\begin{cases} dY_{\tilde{t}} = [-\nabla V(Y_{\tilde{t}}) + 2\beta v_{\tilde{t}} - 2\beta \nabla \log p^{(1)}(v_{\tilde{t}}, Y_{\tilde{t}}, \tilde{t})]d\tilde{t} + \sqrt{2\beta}dW_{\tilde{t}} \\ dv_{\tilde{t}} = [\mu_\theta(Y_{\tilde{t}}) - \sigma_\theta(Y_{\tilde{t}})\nabla \sigma_\theta(Y_{\tilde{t}}) - \sigma_\theta^2(Y_{\tilde{t}})\nabla \log p^{(2)}(v_{\tilde{t}}, Y_{\tilde{t}}, \tilde{t})]d\tilde{t} + \sigma_\theta(Y_{\tilde{t}})d\check{W}_{\tilde{t}} \end{cases}, \quad (85)$$

where  $\nabla \log p^{(j)}(v_{\tilde{t}}, Y_{\tilde{t}}, \tilde{t})$ ,  $j = 1, 2$  stands for  $j$ th component of  $\nabla \log p(v_{\tilde{t}}, Y_{\tilde{t}}, \tilde{t})$ , each of which settles in  $\mathbb{R}^d$ .

*Proof.*

$$d \begin{pmatrix} Y_{\tilde{t}} \\ v_{\tilde{t}} \end{pmatrix} = \left[ \begin{pmatrix} -\nabla V(Y_{\tilde{t}}) + 2\beta v_{\tilde{t}} \\ \mu_{\theta}(Y_{\tilde{t}}) \end{pmatrix} - \right. \quad (86)$$

$$\left. \operatorname{div} \begin{pmatrix} 2\beta & 0 \\ 0 & \sigma_{\theta}^2(Y_{\tilde{t}}) \end{pmatrix} - \begin{pmatrix} 2\beta & 0 \\ 0 & \sigma_{\theta}^2(Y_{\tilde{t}}) \end{pmatrix} \nabla \log p \left( \begin{pmatrix} Y_{\tilde{t}} \\ v_{\tilde{t}} \end{pmatrix}, \tilde{t} \right) \right] d\tilde{t} \quad (87)$$

$$+ \begin{pmatrix} \sqrt{2\beta} dW_{\tilde{t}} \\ \sigma_{\theta}(Y_{\tilde{t}}) d\check{W}_{\tilde{t}} \end{pmatrix} \quad (88)$$

$$= \left[ \begin{pmatrix} -\nabla V(Y_{\tilde{t}}) + 2\beta v_{\tilde{t}} \\ \mu_{\theta}(Y_{\tilde{t}}) \end{pmatrix} - \right. \quad (89)$$

$$\left. \operatorname{div} \begin{pmatrix} 2\beta & 0 \\ 0 & \sigma_{\theta}^2(Y_{\tilde{t}}) \end{pmatrix} - \begin{pmatrix} 2\beta & 0 \\ 0 & \sigma_{\theta}^2(Y_{\tilde{t}}) \end{pmatrix} \begin{pmatrix} \nabla \log p^{(1)}(v_{\tilde{t}}, Y_{\tilde{t}}, \tilde{t}) \\ \nabla \log p^{(2)}(v_{\tilde{t}}, Y_{\tilde{t}}, \tilde{t}) \end{pmatrix} \right] d\tilde{t} \quad (90)$$

$$+ \begin{pmatrix} \sqrt{2\beta} dW_{\tilde{t}} \\ \sigma_{\theta}(Y_{\tilde{t}}) d\check{W}_{\tilde{t}} \end{pmatrix} \quad (91)$$

$$= \begin{pmatrix} -\nabla V(Y_{\tilde{t}}) + 2\beta v_{\tilde{t}} - 2\beta \nabla \log p^{(1)}(v_{\tilde{t}}, Y_{\tilde{t}}, \tilde{t}) \\ \mu_{\theta}(Y_{\tilde{t}}) - 2\sigma_{\theta}(Y_{\tilde{t}}) \nabla \sigma_{\theta}(Y_{\tilde{t}}) - \sigma_{\theta}^2(Y_{\tilde{t}}) \nabla \log p^{(2)}(v_{\tilde{t}}, Y_{\tilde{t}}, \tilde{t}) \end{pmatrix} d\tilde{t} \quad (92)$$

$$+ \begin{pmatrix} \sqrt{2\beta} dW_{\tilde{t}} \\ \sigma_{\theta}(Y_{\tilde{t}}) d\check{W}_{\tilde{t}} \end{pmatrix}, \quad (93)$$

which completes our proof.  $\square$

Parameterize  $\nabla \log p(v_{\tilde{t}}, Y_{\tilde{t}}, \tilde{t})$  by  $s_{\varpi}(v_{\tilde{t}}, Y_{\tilde{t}}, \tilde{t})$ , we have

$$\begin{cases} dY_{\tilde{t}} = [-\nabla V(Y_{\tilde{t}}) + 2\beta v_{\tilde{t}} - 2\beta s_{\varpi}^{(1)}(v_{\tilde{t}}, Y_{\tilde{t}}, \tilde{t})] d\tilde{t} + \sqrt{2\beta} dW_{\tilde{t}} \\ dv_{\tilde{t}} = [\mu_{\theta}(Y_{\tilde{t}}) - 2\sigma_{\theta}(Y_{\tilde{t}}) \nabla \sigma_{\theta}(Y_{\tilde{t}}) - \sigma_{\theta}^2(Y_{\tilde{t}}) s_{\varpi}^{(2)}(v_{\tilde{t}}, Y_{\tilde{t}}, \tilde{t})] d\tilde{t} + \sigma_{\theta}(Y_{\tilde{t}}) d\check{W}_{\tilde{t}} \end{cases}, \quad (94)$$

which will be trained by flow matching[26] or trajectory balance[28].

## 4.2 Continuous Generalization of SAC and GFN

### 4.2.1 Continuous Generalization of SAC

We derive the continuous generalization of SAC from the framework of  $q$ -learning[20], using the HJB equation.

The value function from time  $t$  is given by

$$J(t, y; \pi) = \mathbb{E}_{\mathbb{H}} \left[ \int_t^T \left[ \frac{1}{4\beta} |v_s|^2 - \gamma \log \pi(v_s | Y_s) \right] ds + g(Y_T) \Big| Y_t = y \right], \quad (95)$$

where  $\gamma$  is the exploration-exploitation ratio, and  $\mathbb{E}_{\mathbb{H}}$  is taken concerning both  $Y_t$  and  $\pi$ .

Next, we define  $Q_{\Delta t}(t, y, v; \pi)$  function as the following

$$Q_{\Delta t}(t, y, v; \pi) = \mathbb{E}_{\mathbb{H}} \left[ \int_t^{t+\Delta t} \frac{1}{4\beta} |v|^2 ds + \int_{t+\Delta t}^T \left[ \frac{1}{4\beta} |v_s|^2 - \gamma \log \pi(v_s | Y_s^v) \right] ds + g(Y_T^v) \Big| Y_t^v = y \right], \quad (96)$$

which represents the value of action  $v$  lasts  $\Delta t$  from time  $t$  at state  $y$ . Here, we use superscript  $v$  to indicate variables perturbed by  $v$ .

Taking limit, we have[20, Propostion 3]

**Proposition 4.5.** *The continuous generalization of  $Q$  function, namely  $q$  function in the literature, is given by*

$$q(t, y, v; \pi) := \lim_{\Delta t \rightarrow 0} \frac{Q_{\Delta t}(t, y, v; \pi) - J(t, y; \pi)}{\Delta t} \quad (97)$$

$$= \frac{\partial J}{\partial t}(t, y; \pi) + H(t, y, v, \frac{\partial J}{\partial y}(t, y; \pi), \frac{\partial^2 J}{\partial y^2}(t, y; \pi)), \quad (98)$$

where  $H$  is the Hamiltonian, defined as

$$H(y, v, \mathbf{p}, \mathbf{q}) = [-\nabla V(y) + 2\beta v] \cdot \mathbf{p} + \beta \cdot \mathbf{q} + \frac{1}{4\beta} |v|^2. \quad (99)$$

*Proof.*

$$Q_{\Delta t}(t, y, v; \pi) = \mathbb{E}_{\mathbb{H}} \left[ \int_t^{t+\Delta t} \frac{1}{4\beta} |v|^2 ds + \int_{t+\Delta t}^T \left[ \frac{1}{4\beta} |v_s|^2 - \gamma \log \pi(v_s | Y_s^v) \right] ds + g(Y_T^v) | Y_t^v = y \right] \quad (100)$$

$$= \mathbb{E}_{\mathbb{H}} \left[ \int_t^{t+\Delta t} \frac{1}{4\beta} |v|^2 ds + \mathbb{E}_{\mathbb{H}} \left[ \int_{t+\Delta t}^T \left[ \frac{1}{4\beta} |v_s|^2 - \gamma \log \pi(v_s | Y_s^v) \right] ds + g(Y_T^v) | Y_{t+\Delta t}^v \right] | Y_t^v = y \right] \quad (101)$$

$$= \mathbb{E}_{\mathbb{Q}} \left[ \int_t^{t+\Delta t} \frac{1}{4\beta} |v|^2 ds + J(t + \Delta t, Y_{t+\Delta t}^v; \pi) | Y_t^v = y \right] \quad (102)$$

$$= \mathbb{E}_{\mathbb{Q}} \left[ \int_t^{t+\Delta t} \frac{1}{4\beta} |v|^2 ds + J(t + \Delta t, Y_{t+\Delta t}^v; \pi) - J(t, Y_t^v; \pi) | Y_t^v = y \right] + J(t, Y_t^v; \pi) \quad (103)$$

$$= \mathbb{E}_{\mathbb{Q}} \left[ \int_t^{t+\Delta t} \left[ \frac{\partial J}{\partial t}(s, Y_s^v; \pi) + H(s, Y_s^v, v, \frac{\partial J}{\partial y}(s, Y_s^v; \pi), \frac{\partial^2 J}{\partial y^2}(s, Y_s^v; \pi)) \right] ds | Y_t^v = y \right] + J(t, y; \pi) \quad (104)$$

$$= J(t, y; \pi) + \left[ \frac{\partial J}{\partial t}(s, Y_s^v; \pi) + H(s, Y_s^v, v, \frac{\partial J}{\partial y}(s, Y_s^v; \pi), \frac{\partial^2 J}{\partial y^2}(s, Y_s^v; \pi)) \right] \Delta t + o(\Delta t). \quad (105)$$

which completes the proof.  $\square$

The following theorem establishes that the value function  $J$  will be improved by minimizing the relative entropy between  $\pi$  and  $\exp(\frac{1}{\gamma}H)$

**Theorem 4.6.** *Given  $(t, y) \in [0, T] \times \mathbb{R}^d$ , if two policies  $\pi, \pi' \in \mathcal{P}$ , if*

$$Ent(\pi'(\cdot | t, y) | \exp(\frac{1}{\gamma}H(t, y, \cdot, \frac{\partial J}{\partial y}(t, y; \pi), \frac{\partial^2 J}{\partial y^2}(t, y; \pi)))) \leq Ent(\pi(\cdot | t, y) | \exp(\frac{1}{\gamma}H(t, y, \cdot, \frac{\partial J}{\partial y}(t, y; \pi), \frac{\partial^2 J}{\partial y^2}(t, y; \pi))))), \quad (106)$$

then

$$J(t, y; \pi') \geq J(t, y; \pi). \quad (107)$$

*Proof.* See [20, Theorem 10].  $\square$

**Remark 4.7.** *This theorem does not guarantee the policy  $\pi$  to converge to the optimal one  $\pi^*$ , which is also the case in discretized SAC algorithm[16].*

When we reweigh the probability of actions(at a time) given  $t, y, \pi$ , the value of  $\frac{\partial J}{\partial t}(t, y; \pi)$  is fixed, we have

$$\exp(\frac{1}{\gamma}H(t, y, \cdot, \frac{\partial J}{\partial y}(t, y; \pi), \frac{\partial^2 J}{\partial y^2}(t, y; \pi))) \propto \exp(\frac{1}{\gamma}q(t, y, v; \pi)) \quad (108)$$

Hence, an alternative is to update by

$$Ent(\pi'(\cdot | t, y) | \exp(\frac{1}{\gamma}q(t, y, v; \pi))) \leq Ent(\pi(\cdot | t, y) | \exp(\frac{1}{\gamma}q(t, y, v; \pi))). \quad (109)$$

However, we will use  $H$  to avoid the additional computational cost induced by  $\frac{\partial J}{\partial t}$ .

According to our previous parameterization

$$\pi_F^\theta(v_t | Y_t) = \frac{1}{\sqrt{(2\pi h \sigma_\theta^2(Y_t))^d}} \exp\left(-\frac{1}{2h \sigma_\theta^2(Y_t)}(v_t - \mu_\theta(Y_t))^T(v_t - \mu_\theta(Y_t))\right). \quad (110)$$

We derive the optimality condition as the following

$$\frac{\partial}{\partial \theta} \text{Ent}(\pi_F^\theta | \exp(\frac{1}{\gamma} H)) = \frac{\partial}{\partial \theta} \int_{\mathcal{A}} [\log \pi_F^\theta(v|t, y) - \frac{1}{\gamma} H(t, y, v, \frac{\partial J}{\partial y}(t, y; \pi_F^\theta), \frac{\partial^2 J}{\partial y^2}(t, y; \pi_F^\theta))] \pi_F^\theta(v|t, y) dv \quad (111)$$

$$= \int_{\mathcal{A}} [\log \pi_F^\theta(v|t, y) - \frac{1}{\gamma} H(t, y, v, \frac{\partial J}{\partial y}(t, y; \pi_F^\theta), \frac{\partial^2 J}{\partial y^2}(t, y; \pi_F^\theta))] \frac{\partial \pi_F^\theta(v|t, y)}{\partial \theta} dv \quad (112)$$

$$+ \int_{\mathcal{A}} \frac{\partial}{\partial \theta} (\log \pi_F^\theta(v|t, y)) \pi_F^\theta(v|t, y) dv \quad (113)$$

$$= \int_{\mathcal{A}} [\log \pi_F^\theta(v|t, y) - \frac{1}{\gamma} H(t, y, v, \frac{\partial J}{\partial y}(t, y; \pi_F^\theta), \frac{\partial^2 J}{\partial y^2}(t, y; \pi_F^\theta))] \frac{\partial \pi_F^\theta(v|t, y)}{\partial \theta} dv \quad (114)$$

$$+ \int_{\mathcal{A}} \frac{\partial}{\partial \theta} \pi_F^\theta(v|t, y) dv \quad (115)$$

$$= \int_{\mathcal{A}} [\log \pi_F^\theta(v|t, y) - \frac{1}{\gamma} H(t, y, v, \frac{\partial J}{\partial y}(t, y; \pi_F^\theta), \frac{\partial^2 J}{\partial y^2}(t, y; \pi_F^\theta))] \frac{\partial \pi_F^\theta(v|t, y)}{\partial \theta} dv \quad (116)$$

$$+ \frac{\partial}{\partial \theta} \int_{\mathcal{A}} \pi_F^\theta(v|t, y) dv \quad (117)$$

$$= \int_{\mathcal{A}} [\log \pi_F^\theta(v|t, y) - \frac{1}{\gamma} H(t, y, v, \frac{\partial J}{\partial y}(t, y; \pi_F^\theta), \frac{\partial^2 J}{\partial y^2}(t, y; \pi_F^\theta))] \frac{\partial \pi_F^\theta(v|t, y)}{\partial \theta} dv \quad (118)$$

$$= \int_{\mathcal{A}} [\log \pi_F^\theta(v|t, y) - \frac{1}{\gamma} H(t, y, v, \frac{\partial J}{\partial y}(t, y; \pi_F^\theta), \frac{\partial^2 J}{\partial y^2}(t, y; \pi_F^\theta))] \frac{\partial}{\partial \theta} (\log \pi_F^\theta(v|t, y)) \pi_F^\theta(v|t, y) dv. \quad (119)$$

We have

$$\begin{aligned} \log \pi_F^\theta(v_t|Y_t) &= -\frac{d}{2} \log(2\pi h) - d \log \sigma_\theta(Y_t) - \frac{1}{2\sigma_\theta^2(Y_t)} (v_t - \mu_\theta(Y_t))^T (v_t - \mu_\theta(Y_t)) \\ \implies \frac{\partial}{\partial \theta} \log \pi_F^\theta(v_t|Y_t) &= -\frac{d}{\sigma_\theta(Y_t)} \frac{\partial \sigma_\theta(Y_t)}{\partial \theta} + \frac{(v_t - \mu_\theta(Y_t))^T (v_t - \mu_\theta(Y_t))}{3\sigma_\theta^3(Y_t)} \frac{\partial \sigma_\theta(Y_t)}{\partial \theta} + \frac{1}{\sigma_\theta^2(Y_t)} (v_t - \mu_\theta(Y_t))^T \frac{\partial \mu_\theta(Y_t)}{\partial \theta} \end{aligned}$$

If an accurate approximation of  $H(t, y, \cdot, \frac{\partial J}{\partial y}(t, y; \pi), \frac{\partial^2 J}{\partial y^2}(t, y; \pi))$  is available, we can improve current policy  $\pi_F^\theta$  by sample random actions  $v_t$  from it and take gradient steps, either offline or online

$$\theta \leftarrow \theta - [\log \pi_F^\theta(v_t|t, Y_t) - \frac{1}{\gamma} H(t, Y_t, v_t, \frac{\partial J}{\partial y}(t, Y_t; \pi_F^\theta), \frac{\partial^2 J}{\partial y^2}(t, Y_t; \pi_F^\theta))] \frac{\partial}{\partial \theta} (\log \pi_F^\theta(v_t|t, Y_t))^T \quad (120)$$

where the explicit formula of  $\frac{\partial}{\partial \theta} \log \pi_F^\theta(v_t|Y_t)$  is given above.

Next, we derive how to get an satisfying approximation of  $H(t, Y_t, v_t, \frac{\partial J}{\partial y}(t, Y_t; \pi), \frac{\partial^2 J}{\partial y^2}(t, Y_t; \pi))$ . Instead of approximating  $q$  from  $Q$  by discretization or defining adjoint state, we directly compute Hamiltonian using the Girsanov reparameterization trick in [10, Appendix C.2, Proposition 4]. We will slightly generalize the diffusion term of the original proposition in the diffusion term from time-dependent-only to time-space-dependent. The idea of proof is roughly the same.

**Theorem 4.8.** *Consider diffusion process*

$$dX_s = b(X_s, s)ds + \sigma(X_s, s)dW_s, X_0 = x \in \mathbb{R}^d \quad (121)$$

Given reparameterization flow  $Z : \mathbb{R}^d \times [0, T] \xrightarrow{C^2} \mathbb{R}^d$ , satisfying  $Z(z, 0) = z, \forall z \in \mathbb{R}^d$  and  $Z(0, s) = 0, \forall s \in [0, T]$ . Let  $F : C([0, T]; \mathbb{R}^d) \rightarrow \mathbb{R}^d$  be a Frechet-differentiable functional, we have

$$\nabla_x \mathbb{E}[F(X)|X_0 = x] = \mathbb{E}[\nabla_z F(X + Z(z, \cdot))]|_{z=0} \quad (122)$$

$$+ F(X) \exp \int_0^T (\nabla_z Z(z, s)|_{z=0} \nabla_x b(X_s^x, s) - \nabla_z \partial_s Z(z, s)|_{z=0}) (\sigma(X_s^x, s)^{-1})^T dW_t] \quad (123)$$

where we use capital  $X$  without time subscript and  $Z(z, \cdot)$  to denote one complete trajectory generated by the corresponding process.

**Remark 4.9.** *The key idea of Girsanov reparameterization is similar to  $\frac{d}{dx}f(x)|_{x=x_0} = \frac{d}{dt}f(x_0+t)|_{t=0}$ , that is, shift the gradient of  $x$  to a new parameter  $t$ . Integrating by time,  $t$  becomes a family of reparameterization  $C^2$  section  $Z(z, s)$ , which connects to bundle theory and Lie algebra in differential geometry. We will further develop this idea in sections later.*

*Proof.* Observe that

$$\nabla_x \mathbb{E}[F(X)|X_0 = x] = \nabla_z \mathbb{E}[F(X)|X_0 = x + z]|_{z=0}. \quad (124)$$

Our next target is to shift the perturbed process  $X_t$  to its original position to separate the effect of perturbation, which is exactly the reparameterization component.

Define shifted process

$$dX_s^{x+z} := b(X_s^{x+z}, s)ds + \sigma(X_t^{x+z}, t)dW_t, X_0^{x+z} = x + z, \quad (125)$$

and shifted-back process

$$d\check{X}_s^x := [b(\check{X}_s^x + Z(z, s), s) - \partial_s Z(z, s)]ds + \sigma(\check{X}_s^x + Z(z, s))dW_t, d\check{X}_0^x = x \quad (126)$$

We have

$$d(\check{X}_s^x + Z(z, s)) = b(\check{X}_s^x + Z(z, s), s)ds + \sigma(\check{X}_s^x + Z(z, s))dW_t, \check{X}_0^x + Z(z, 0) = x + z, \quad (127)$$

by the uniqueness of the strong solution[29, Theorem 5.2.1], this formula explicitly reparameterizes the shifted process for any  $z \in \mathbb{R}^d$ . Thus

$$\mathbb{E}[F(X)|X_0 = x + z] = \mathbb{E}[F(X^{x+z})] \quad (128)$$

$$= \mathbb{E}[F(\check{X}^x + Z(z, \cdot))] \quad (129)$$

$$= \mathbb{E}[F(X + Z(z, \cdot)) \exp(\int_0^T \sigma(X_s^x, s)^{-1}(b(X_s^x + Z(z, s), s) - b(X_s^x, s) - \partial_s Z(z, s))dW_s) \quad (130)$$

$$- \frac{1}{2} \int_0^T |\sigma(X_s^x, s)^{-1}(b(X_s^x + Z(z, s), s) - b(X_s^x, s) - \partial_s Z(z, s))|^2 ds)]. \quad (131)$$

Plug it in the representation of the gradient

$$\nabla_z \mathbb{E}[F(X)|X_0 = x + z] = \nabla_z \mathbb{E}[F(X + Z(z, \cdot)) \exp(\int_0^T \sigma(X_s^x, s)^{-1}(b(X_s^x + Z(z, s), s) - b(X_s^x, s) - \partial_s Z(z, s))dW_s) \quad (132)$$

$$- \frac{1}{2} \int_0^T |\sigma(X_s^x, s)^{-1}(b(X_s^x + Z(z, s), s) - b(X_s^x, s) - \partial_s Z(z, s))|^2 ds)] \quad (133)$$

We have

$$\nabla_z \mathbb{E}[F(X)|X_0 = x + z] = \mathbb{E}[\nabla_z F(X + Z(z, \cdot)) \exp(\int_0^T \sigma(X_s^x, s)^{-1}(b(X_s^x + Z(z, s), s) - b(X_s^x, s) - \partial_s Z(z, s))dW_s) \quad (134)$$

$$- \frac{1}{2} \int_0^T |\sigma(X_s^x, s)^{-1}(b(X_s^x + Z(z, s), s) - b(X_s^x, s) - \partial_s Z(z, s))|^2 ds) \quad (135)$$

$$+ F(X + Z(z, \cdot)) \exp(\int_0^T \sigma(X_s^x, s)^{-1}(b(X_s^x + Z(z, s), s) - b(X_s^x, s) - \partial_s Z(z, s))dW_s) \quad (136)$$

$$- \frac{1}{2} \int_0^T |\sigma(X_s^x, s)^{-1}(b(X_s^x + Z(z, s), s) - b(X_s^x, s) - \partial_s Z(z, s))|^2 ds) \quad (137)$$

$$\cdot (\nabla_z \int_0^T \sigma(X_s^x, s)^{-1}(b(X_s^x + Z(z, s), s) - b(X_s^x, s) - \partial_s Z(z, s))dW_s) \quad (138)$$

$$- \frac{1}{2} \nabla_z \int_0^T |\sigma(X_s^x, s)^{-1}(b(X_s^x + Z(z, s), s) - b(X_s^x, s) - \partial_s Z(z, s))|^2 ds)] \quad (139)$$

Notice

$$\nabla_z \int_0^T |\sigma(X_s^x, s)^{-1}(b(X_s^x + Z(z, s), s) - b(X_s^x, s) - \partial_s Z(z, s))|^2 ds|_{z=0} \quad (140)$$

$$= \int_0^T \nabla_z |\sigma(X_s^x, s)^{-1}(b(X_s^x + Z(z, s), s) - b(X_s^x, s) - \partial_s Z(z, s))|^2|_{z=0} ds \quad (141)$$

$$= \int_0^T 2(b(X_s^x + Z(0, s), s) - b(X_s^x, s) - \partial_s Z(0, s))^T (\sigma(X_s^x, s)^{-1})^T \nabla_z (\cdots)|_{z=0} ds \quad (142)$$

$$= 0, \quad (143)$$

and

$$\nabla_z \int_0^T \sigma(X_s^x, s)^{-1}(b(X_s^x + Z(z, s), s) - b(X_s^x, s) - \partial_s Z(z, s)) dW_s \quad (144)$$

$$= \int_0^T (\nabla_z Z(z, s)|_{z=0} \nabla_x b(X_s^x, s) - \nabla_z \partial_s Z(z, s)|_{z=0}) (\sigma(X_s^x, s)^{-1})^T dW_t \quad (145)$$

Plug them in, we have

$$\nabla_x \mathbb{E}[F(X)|X_0 = x] = \nabla_z \mathbb{E}[F(X)|X_0 = x + z]|_{z=0} \quad (146)$$

$$= \mathbb{E}[\nabla_z F(X + Z(z, \cdot))]|_{z=0} \quad (147)$$

$$+ F(X) \exp \int_0^T (\nabla_z Z(z, s)|_{z=0} \nabla_x b(X_s^x, s) - \nabla_z \partial_s Z(z, s)|_{z=0}) (\sigma(X_s^x, s)^{-1})^T dW_t], \quad (148)$$

which completes the proof.  $\square$

Apply this formula on the  $\frac{\partial J}{\partial y}$ , we have

$$\begin{aligned} J(t, y; \pi_F^\theta) &= \mathbb{E}_{\mathbb{H}} \left[ \int_t^T \left[ \frac{1}{4\beta} |v_s|^2 - \gamma \log \pi(v_s | Y_s) \right] ds + g(Y_T) \Big| Y_t = y \right] \\ \implies \frac{\partial J}{\partial y}(t, y; \pi_F^\theta) &= \frac{\partial}{\partial y} \mathbb{E}_{\pi_F^\theta} [\mathbb{E}_{\mathbb{Q}} \left[ \int_t^T \left[ \frac{1}{4\beta} |v_s|^2 - \gamma \log \pi(v_s | Y_s) \right] ds + g(Y_T) \Big| Y_t = y \right]] \\ &= \mathbb{E}_{\pi_F^\theta} [\nabla_y \mathbb{E}_{\mathbb{Q}} \left[ \int_t^T \left[ \frac{1}{4\beta} |v_s|^2 - \gamma \log \pi(v_s | Y_s) \right] ds + g(Y_T) \Big| Y_t = y \right]]. \end{aligned}$$

From reparameterization trick

$$\nabla_z \int_t^T \left[ \frac{1}{4\beta} |\mu_\theta(Y_s + Z(z, s)) + \sigma_\theta(Y_s + Z(z, s)) \xi|^2 - \gamma \left( -\frac{d}{2} \log(2\pi h) \right) - d \log \sigma_\theta(Y_s + Z(z, s)) - \frac{1}{2} \xi^T \xi \right] ds \quad (149)$$

$$+ \nabla_z g(Y_T + Z(z, T)) \quad (150)$$

$$= \int_t^T \left( \frac{1}{2\beta} (\mu_\theta(Y_s + Z(z, s)) + \sigma_\theta(Y_s + Z(z, s)) \xi)^T \left( \frac{\partial \mu_\theta}{\partial y}(Y_s + Z(z, s)) \frac{\partial Z}{\partial z}(z, s) \right) \right) ds \quad (151)$$

$$+ \xi \frac{\partial \sigma_\theta}{\partial y}(Y_s + Z(z, s)) \frac{\partial Z}{\partial z}(z, s) + \frac{d\gamma}{\sigma_\theta(Y_s + Z(z, s))} \frac{\partial \sigma_\theta}{\partial y}(Y_s + Z(z, s)) \frac{\partial Z}{\partial z}(z, s) ds \quad (152)$$

$$+ \frac{\partial g}{\partial y}(Y_T + Z(z, T)) \frac{\partial Z}{\partial z}(z, T). \quad (153)$$

Set  $z = 0$ , we have

$$\nabla_z \int_t^T \left[ \frac{1}{4\beta} |\mu_\theta(Y_s + Z(z, s)) + \sigma_\theta(Y_s + Z(z, s))\xi|^2 - \gamma \left(-\frac{d}{2} \log(2\pi h)\right) - d \log \sigma_\theta(Y_s + Z(z, s)) - \frac{1}{2} \xi^T \xi \right] ds \quad (154)$$

$$+ \nabla_z g(Y_T + Z(z, T)) \Big|_{z=0} \quad (155)$$

$$= \int_t^T \left( \frac{1}{2\beta} (\mu_\theta(Y_s) + \sigma_\theta(Y_s)\xi)^T \left( \frac{\partial \mu_\theta}{\partial y}(Y_s) \frac{\partial Z}{\partial z}(0, s) \right) \right) \quad (156)$$

$$+ \xi \frac{\partial \sigma_\theta}{\partial y}(Y_s) \frac{\partial Z}{\partial z}(0, s) + \frac{d\gamma}{\sigma_\theta(Y_s)} \frac{\partial \sigma_\theta}{\partial y}(Y_s) \frac{\partial Z}{\partial z}(0, s) ds \quad (157)$$

$$+ \frac{\partial g}{\partial y}(Y_T) \frac{\partial Z}{\partial z}(0, T) \quad (158)$$

$$= \int_t^T \left( \frac{1}{2\beta} (\mu_\theta(Y_s) + \sigma_\theta(Y_s)\xi)^T \frac{\partial \mu_\theta}{\partial y}(Y_s) + \xi \frac{\partial \sigma_\theta}{\partial y}(Y_s) + \frac{d\gamma}{\sigma_\theta(Y_s)} \frac{\partial \sigma_\theta}{\partial y}(Y_s) \right) \frac{\partial Z}{\partial z}(0, s) ds \quad (159)$$

$$+ \frac{\partial g}{\partial y}(Y_T) \frac{\partial Z}{\partial z}(0, T) \quad (160)$$

The second term is easier to compute

$$\left( \int_t^T \left( \frac{1}{4\beta} |\mu_\theta(Y_s) + \sigma_\theta(Y_s)\xi|^2 - \gamma \left(-\frac{d}{2} \log(2\pi h)\right) - d \log \sigma_\theta(Y_s) - \frac{1}{2} \xi^T \xi \right) ds \right. \quad (161)$$

$$\left. + g(Y_T) \exp \int_0^T \left( \frac{\partial Z}{\partial z}(0, s) (-\nabla^2 V(Y_s) + 2\beta \left( \frac{\partial \mu_\theta}{\partial y}(Y_s) + \xi^T \frac{\partial \sigma_\theta}{\partial y}(Y_s) \right)) - \frac{\partial^2 Z}{\partial z \partial s}(0, s) \right) dW_t \right) \quad (162)$$

Combining all of them

$$\nabla_y \mathbb{E}_\mathbb{Q} \left[ \int_t^T \left[ \frac{1}{4\beta} |v_s|^2 - \gamma \log \pi(v_s | Y_s) \right] ds + g(Y_T) \Big| Y_t = y \right] \quad (163)$$

$$= \mathbb{E}_\mathbb{Q} \left[ \int_t^T \left( \frac{1}{2\beta} (\mu_\theta(Y_s) + \sigma_\theta(Y_s)\xi)^T \frac{\partial \mu_\theta}{\partial y}(Y_s) + \xi \frac{\partial \sigma_\theta}{\partial y}(Y_s) + \frac{d\gamma}{\sigma_\theta(Y_s)} \frac{\partial \sigma_\theta}{\partial y}(Y_s) \right) \frac{\partial Z}{\partial z}(0, s) ds \right. \quad (164)$$

$$\left. + \frac{\partial g}{\partial y}(Y_T) \frac{\partial Z}{\partial z}(0, T) + \left( \int_t^T \left( \frac{1}{4\beta} |\mu_\theta(Y_s) + \sigma_\theta(Y_s)\xi|^2 - \gamma \left(-\frac{d}{2} \log(2\pi h)\right) - d \log \sigma_\theta(Y_s) - \frac{1}{2} \xi^T \xi \right) ds \right. \quad (165)$$

$$\left. + g(Y_T) \exp \int_0^T \left( \frac{\partial Z}{\partial z}(0, s) (-\nabla^2 V(Y_s) + 2\beta \left( \frac{\partial \mu_\theta}{\partial y}(Y_s) + \xi^T \frac{\partial \sigma_\theta}{\partial y}(Y_s) \right)) - \frac{\partial^2 Z}{\partial z \partial s}(0, s) \right) dW_t \Big| Y_t = y \right] \quad (166)$$

Thus

$$\frac{\partial J}{\partial y}(t, y; \pi_F^\theta) = \mathbb{E}_\mathbb{H} \left[ \int_t^T \left( \frac{1}{2\beta} (\mu_\theta(Y_s) + \sigma_\theta(Y_s)\xi)^T \frac{\partial \mu_\theta}{\partial y}(Y_s) + \xi \frac{\partial \sigma_\theta}{\partial y}(Y_s) + \frac{d\gamma}{\sigma_\theta(Y_s)} \frac{\partial \sigma_\theta}{\partial y}(Y_s) \right) \frac{\partial Z}{\partial z}(0, s) ds \right. \quad (167)$$

$$\left. + \frac{\partial g}{\partial y}(Y_T) \frac{\partial Z}{\partial z}(0, T) + \left( \int_t^T \left( \frac{1}{4\beta} |\mu_\theta(Y_s) + \sigma_\theta(Y_s)\xi|^2 - \gamma \left(-\frac{d}{2} \log(2\pi h)\right) - d \log \sigma_\theta(Y_s) - \frac{1}{2} \xi^T \xi \right) ds \right. \quad (168)$$

$$\left. + g(Y_T) \exp \int_0^T \left( \frac{\partial Z}{\partial z}(0, s) (-\nabla^2 V(Y_s) + 2\beta \left( \frac{\partial \mu_\theta}{\partial y}(Y_s) + \xi^T \frac{\partial \sigma_\theta}{\partial y}(Y_s) \right)) - \frac{\partial^2 Z}{\partial z \partial s}(0, s) \right) dW_t \Big| Y_t = y \right] \quad (169)$$

Suppose the reparameterization section  $Z$  is already selected. To distinguish the original expression of  $\frac{\partial J}{\partial y}$  from its reparameterized form, we denote the latter one  $L_Z J$ , and parameterize it as  $L_Z^\psi J$  (this notation is taken from that of Lie derivative in differential geometry).

Consider  $L_Z$  loss

$$\mathcal{L}_\psi(t, y, \theta) := |L_Z^\psi J(t, y, \theta) - L_Z J(t, y, \pi)|^2 \quad (170)$$

$$\leq \mathbb{E}_\mathbb{H} |L_Z^\psi J(t, y, \theta) - \int_t^T (\frac{1}{2\beta}(\mu_\theta(Y_s) + \sigma_\theta(Y_s)\xi)^T \frac{\partial \mu_\theta}{\partial y}(Y_s) + \xi \frac{\partial \sigma_\theta}{\partial y}(Y_s) + \frac{d\gamma}{\sigma_\theta(Y_s)} \frac{\partial \sigma_\theta}{\partial y}(Y_s)) \frac{\partial Z}{\partial z}(0, s) ds| \quad (171)$$

$$+ \frac{\partial g}{\partial y}(Y_T) \frac{\partial Z}{\partial z}(0, T) + (\int_t^T (\frac{1}{4\beta}|\mu_\theta(Y_s) + \sigma_\theta(Y_s)\xi|^2 - \gamma(-\frac{d}{2}\log(2\pi h)) - d\log\sigma_\theta(Y_s) - \frac{1}{2}\xi^T \xi) ds \quad (172)$$

$$+ g(Y_T)) \exp \int_0^T (\frac{\partial Z}{\partial z}(0, s)(-\nabla^2 V(Y_s) + 2\beta(\frac{\partial \mu_\theta}{\partial y}(Y_s) + \xi^T \frac{\partial \sigma_\theta}{\partial y}(Y_s))) - \frac{\partial^2 Z}{\partial z \partial s}(0, s)) dW_t|^2, \quad (173)$$

which can be computed by simulation in online setting or by replaying in offline setting. The expectation  $\mathbb{E}_\mathbb{H}$  can be approximated batch-wise, and we will denote the corresponding estimator  $\hat{\mathcal{L}}_\psi(t, y, \theta)$ .

**Remark 4.10.** *Note that the value function does not depend on any specific  $v$ , but on the average behavior of  $\pi$  along the trajectories sampled.*

This gives us the Hamiltonian on approximated  $L_Z J$

$$H(t, y, v, L_Z^\psi J(t, y, \theta), \nabla_y L_Z^\psi J(t, y, \theta)) = [-\nabla V(y) + 2\beta v] \cdot L_Z^\psi J(t, y, \theta) + \beta \cdot \nabla_y L_Z^\psi J(t, y, \theta) + \frac{1}{4\beta}|v|^2, \quad (174)$$

and the optimization step of policy becomes

$$\theta \leftarrow \theta - [\log \pi_F^\theta(v_t | Y_t) - \frac{1}{\gamma} H(t, Y_t, v_t, L_Z^\psi J(t, Y_t, \theta), \nabla_y L_Z^\psi J(t, Y_t, \theta))] \frac{\partial}{\partial \theta} (\log \pi_F^\theta(v_t | Y_t))^T \quad (175)$$

**Remark 4.11.** *The advantage of Girsanov reparameterization is that it does not involve derivatives of the initial condition, which effectively avoids costly backpropagation along paths when the network is on a large scale, at the expense of affecting the variance of the result by reparameterization section Z[10, Proposition 2, Theorem 2], a new issue for achieving lower mean square error. We will tackle this problem by injecting additional physics information via principal bundle theory.*

Following the structure of its discretized counterparts in algorithm 1, we derive continuous time Soft Actor Critic without any discretization step.

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**Algorithm 3** Continuous Soft Actor-Critic (Conti-SAC) algorithm for SOC problem

---

Initialize policy parameters  $\theta$ , Hamiltonian parameters  $\psi_1, \psi_2$ , empty replay buffer  $\mathcal{D}$ , tradeoff constant  $\gamma$ , target parameter  $\psi_{targ,1} \leftarrow \psi_1, \psi_{targ,2} \leftarrow \psi_2$ , smoothing parameter  $\rho$ , and selected reparameterization section  $Z$ .

**repeat**

- 3: Observe state  $Y_t$  and select action  $v_t$  from distribution  $\pi_F^\theta(\cdot | Y_t)$  throughout  $0 \leq t \leq T$ .  
Store trajectories  $(Y_t, v_t)_{0 \leq t \leq T}$  in replay buffer  $\mathcal{D}$ .

**if** it's time to update **then**

- 6: **for** each gradient step **do**

Randomly sample a batch  $B$  of trajectories  $(Y_t, v_t)_{0 \leq t \leq T}$  from  $\mathcal{D}$ .

For  $t_j \sim \text{Unif}[0, T], 1 \leq j \leq |B|$ , compute the targets for  $L_Z^\psi J$

- 9:

$$y_L^{t_j} := \int_{t_j}^T (\frac{1}{2\beta}(\mu_\theta(Y_s) + \sigma_\theta(Y_s)\xi)^T \frac{\partial \mu_\theta}{\partial y}(Y_s) + \xi \frac{\partial \sigma_\theta}{\partial y}(Y_s) + \frac{d\gamma}{\sigma_\theta(Y_s)} \frac{\partial \sigma_\theta}{\partial y}(Y_s)) \frac{\partial Z}{\partial z}(0, s) ds$$

$$+ \frac{\partial g}{\partial y}(Y_T) \frac{\partial Z}{\partial z}(0, T) + (\int_{t_j}^T (\frac{1}{4\beta}|\mu_\theta(Y_s) + \sigma_\theta(Y_s)\xi|^2 - \gamma(-\frac{d}{2}\log(2\pi h)) - d\log\sigma_\theta(Y_s) - \frac{1}{2}\xi^T \xi) ds$$

$$+ g(Y_T)) \exp \int_0^T (\frac{\partial Z}{\partial z}(0, s)(-\nabla^2 V(Y_s) + 2\beta(\frac{\partial \mu_\theta}{\partial y}(Y_s) + \xi^T \frac{\partial \sigma_\theta}{\partial y}(Y_s))) - \frac{\partial^2 Z}{\partial z \partial s}(0, s)) dW_t$$

Update  $L_Z^\psi J$  by one step of gradient descent

$$\nabla_{\psi_i} \hat{\mathcal{L}}_{\psi_i} = \nabla_{\psi_i} \frac{1}{|B|} \sum_{1 \leq j \leq |B|} \left( L_Z^{\psi_i}(Y_t, v_t, \theta) - y_L^{t_j} \right)^2, \text{ for } i \in \{1, 2\}.$$

12: Compute Hamiltonian and update policy by one step of gradient descent

$$\theta \leftarrow \theta - [\log \pi_F^\theta(v_t|t, Y_t) - \frac{1}{\gamma} H(t, Y_t, v_t, L_Z^\psi J(t, Y_t, \theta), \nabla_y L_Z^\psi J(t, Y_t, \theta))] \frac{\partial}{\partial \theta} (\log \pi_F^\theta(v_t|t, Y_t))^T \quad (176)$$

where the reparametrization trick is used, that is

$$v_t = \mu_\theta(Y_t) + \sigma_\theta(Y_t) \cdot \xi, \quad \xi \sim \mathcal{N}(0, \sqrt{h} I_d). \quad (177)$$

15: Update target network

$$\psi_{\text{targ}, i} \leftarrow \rho \psi_{\text{targ}, i} + (1 - \rho) \psi_i \quad \text{for } i = 1, 2 \quad (178)$$

**end for**

**end if**

18: **until** convergence

#### 4.2.2 Continuous Generalization of GFlowNet

We could consider two generalizations to generalize GFlowNet to continuous adaptation: evaluate the differences in measure along given paths through the Girsanov theorem, a natural extension of trajectory balance to continuous cases, or directly match the flow via  $L_2$  distance through flow matching.

Let us begin with trajectory balance situation. In discrete case, for fixed starting point, the trajectory balance loss of GFN is given by

$$\mathcal{L}_{\text{TB}} = \left( \log \frac{\chi \prod_{t=0}^{T-1} P_F(Y_{t+\Delta t}|Y_t; \theta)}{R(Y_T) \prod_{\tilde{t}=0}^{T-1} P_B(Y_{\tilde{t}+\Delta \tilde{t}}|Y_{\tilde{t}}; \theta, \varpi)} \right)^2 \quad (179)$$

where  $P_F$  and  $P_B$  are given by the transition probability of the SDE system representing the sampling strategy,  $R(Y_T)$  is the reward, satisfying

$$R(Y_T) = \frac{1}{4\beta} \sum_{t=0}^{T-1} \Delta t |v_t|^2 + g(Y_T) \quad (180)$$

and  $\chi$  is the normalizing constant of  $R$  to be learnt.

Consider

$$\log \frac{\chi \prod_{t=0}^{T-1} P_F(Y_{t+\Delta t}|Y_t; \theta)}{R(Y_T) \prod_{\tilde{t}=0}^{T-1} P_B(Y_{\tilde{t}+\Delta \tilde{t}}|Y_{\tilde{t}}; \theta, \varpi)} \quad (181)$$

$$= \log \frac{\chi \cdot \exp \sum_{t=0}^{T-1} \log P_F(Y_{t+\Delta t}|Y_t; \theta)}{R(Y_T) \cdot \exp \sum_{\tilde{t}=0}^{T-1} \log P_B(Y_{\tilde{t}+\Delta \tilde{t}}|Y_{\tilde{t}}; \theta, \varpi)}. \quad (182)$$

Notice that  $\tilde{t} = T - t$ , we have

$$dY_{T-t} = [-\nabla V(Y_{T-t}) + 2\beta v_{T-t} - 2\beta s_\varpi(v_{T-t}, Y_{T-t}, T-t)] d(T-t) + \sqrt{2\beta} dW_{T-t} \quad (183)$$

$$\implies dY_{T-t} = [\nabla V(Y_{T-t}) - 2\beta v_{T-t} + 2\beta s_\varpi(v_{T-t}, Y_{T-t}, T-t)] dt + \sqrt{2\beta} dW_{T-t} \quad (184)$$

followed by

$$\log \chi - \log R(Y_T) + \sum_{t=0}^{T-1} \log \frac{P_F(Y_{t+\Delta t}|Y_t; \theta)}{P_B(Y_t|Y_{t+\Delta t}; \theta, \varpi)} \quad (185)$$

$$= \log \chi - \log R(Y_T) - \sum_{t=0}^{T-1} \log \frac{P_B(Y_t|Y_{t+\Delta t}; \theta, \varpi)}{P_F(Y_{t+\Delta t}|Y_t; \theta)} \quad (186)$$

Notice, when we perform gradient descent, the  $Y_t(\omega)$  here are specified trajectory in the replay buffer.

Taking limit, the equation above correspond to

$$\mathcal{L}_{TB} := \mathbb{E}_{\mathbb{H}}(\log \chi - \log \frac{d\mathbb{M}}{d\mathbb{Q}}(Y(\omega)) - \log R(Y_T))^2 \quad (187)$$

$$\text{where: } dY_t = [-\nabla V(Y_t) + 2\beta v_t]dt + \sqrt{2\beta}dW_t \sim \mathbb{Q}, \quad (188)$$

$$dY_{T-t} = [\nabla V(Y_{T-t}) - 2\beta v_{T-t} + 2\beta s_{\varpi}^{(1)}(v_{T-t}, Y_{T-t}, T-t)]dt + \sqrt{2\beta}dW_{T-t} \sim \mathbb{M}. \quad (189)$$

Suppose a trajectory  $(Y_t, v_t), 0 \leq t \leq T$  is taken from the replay buffer, we have

$$\frac{d\mathbb{M}}{d\mathbb{Q}} = \exp\left(-\int_0^T \frac{2\nabla V(Y_t) - 4\beta v_t + 2\beta s_{\varpi}^{(1)}(v_t, Y_t, T-t)}{\sqrt{2\beta}} dW_s\right) \quad (190)$$

$$-\frac{1}{2} \int_0^T \frac{(2\nabla V(Y_t) - 4\beta v_t + 2\beta s_{\varpi}^{(1)}(v_t, Y_t, T-t))^2}{2\beta} ds, \quad (191)$$

from Girsanov's theorem.

Finally, we have

$$\mathcal{L}_{TB} = \mathbb{E}_{\mathbb{H}}(\log \chi + \int_0^T \frac{2\nabla V(Y_t) - 4\beta v_t + 2\beta s_{\varpi}^{(1)}(v_t, Y_t, T-t)}{\sqrt{2\beta}} dW_s) \quad (192)$$

$$+ \frac{1}{2} \int_0^T \frac{(2\nabla V(Y_t) - 4\beta v_t + 2\beta s_{\varpi}^{(1)}(v_t, Y_t, T-t))^2}{2\beta} ds - \log R(Y_T))^2. \quad (193)$$

**Remark 4.12.** Although we use experience replay in offline training, we do not detach  $v_F^\theta$  from that graph, that is,  $\theta, \varpi, \chi$  are updated simultaneously.

The flow matching loss is easier to compute, but requires additional backward trajectory samples.

$$\mathcal{L}_{FM} = \mathbb{E}_{t \sim [0, T]} \mathbb{E}_{\mathbb{H}} |Y_t^{forward} - Y_t^{backward}|^2. \quad (194)$$

We present the pseudocode for continuous GFN under the flow matching case.

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**Algorithm 4** Continuous Generative Flow Network (Conti-GFN) algorithm for SOC problem(flow matching version)

---

Initialize GFlowNet  $\theta, \varpi$ , partition function  $\chi$  and empty replay buffer  $\mathcal{D}$ .

**repeat**

- Collect a set of paths  $(Y_t^{forward})_{0 \leq t \leq T}$  by interaction with forward policy  $v_F^\theta$ .
- 4: Collect a set of paths  $(Y_t^{backward})_{0 \leq t \leq T}$  by interaction with backward policy  $v_F^\theta, s_{\varpi}$ .

**if** it's time to update **then**

- Get mini-batch  $\mathcal{B}$  sampled from  $\mathcal{D}$
- Sample a batch of update time  $t \sim [0, T]$

- 8: Update GFlowNet

**end if**

**until** convergence

---

$$\hat{\mathcal{L}}_{FM} = \frac{1}{|\mathcal{B}|} \sum_{k=1}^{|\mathcal{B}|} |Y_t^{forward, k} - Y_t^{backward, k}|^2. \quad (195)$$

## 5 Variance Reduction via Principal Bundle Theory

In this section, our main focus lies on how to select a reparameterization flow  $Z(z, t)$  that fully leverages physics information available by principal bundle theory. We present our "physics-informed reparameterization" beginning with an overview of equivariant flow in section 5.1, then we delve into the relationship between reparameterization and variance in section 5.2, and ultimately, we outline our innovative reparameterization method grounded in principal bundle theory in 5.3.

## 5.1 Equivariant Flow

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